

On a problem of A. V. Grishin

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October 7, 2009

Abstract

In this note, we offer a short proof of V. V. Shchigolev's result that over any field k of characteristic $p > 2$, the T -space generated by $x_1^p, x_1^p x_2^p \dots$ is finitely based, which answered a question raised by A. V. Grishin. More precisely, we prove that for any field of any positive characteristic, $R_2^{(d)} = R_3^{(d)}$ for every positive integer d , and that over an infinite field of characteristic $p > 2$, $L_2 = L_3$. Moreover, if the characteristic of k does not divide d , we prove that $R_1^{(d)}$ is an ideal of $k_0\langle X \rangle$ and thus in particular, $R_1^{(d)} = R_2^{(d)}$. Finally, we show that over any field of characteristic $p > 2$, $R_1^{(d)} \neq R_2^{(d)}$ and $L_1 \neq L_2$.

1 Introduction

In [1] (and later in [2], the survey paper with V. V. Shchigolev), A. V. Grishin proved that in the free associative algebra with countably infinite generating set $\{x_1, x_2, \dots\}$ over a field of characteristic 2, the T -space that is generated by the set $\{x_1^2, x_1^2 x_2^2, \dots\}$ is not finitely based, and he raised the question as to whether or not, in the corresponding setting but over a field of characteristic $p > 2$, the T -space generated by $\{x_1^p, x_1^p x_2^p, \dots\}$ is finitely based. This was resolved by V. V. Shchigolev in [3], wherein he proved that over an infinite field of characteristic $p > 2$, this T -space is finitely based. In fact, if we let L_1 denote the T -space generated by $\{x_1^p\}$, and then for each positive integer n , let L_{n+1} denote the T -space generated by $L_n \cup L_n x_{n+1}^p$, Shchigolev proves in [3] that $L_p = L_{p+1}$. To do this, he made use of another family of T -spaces defined in [3] as follows. Let k denote an arbitrary field of characteristic p , let $X = \{x_1, x_2, \dots\}$ be a countably infinite set, and let $k_0\langle X \rangle$ denote the free associative k -algebra over the set X . For each positive integer d , let $S_d(x)$ denote the sum $\sum_{\sigma \in \Sigma_d} \prod_{i=1}^d x_{\sigma(i)}$, where Σ_d is the symmetric group on d letters. Let $R_1^{(d)}$ denote the T -space of $k_0\langle X \rangle$ that is generated by $S_d(x)$, and for each positive integer n , let $R_{n+1}^{(d)}$ denote the T -space of $k_0\langle X \rangle$ that is generated by $R_n^{(d)} \cup R_n^{(d)} S_d(x)$. As a key step in his demonstration that $L_p = L_{p+1}$, Shchigolev proves that for any positive integer d , $R_d^{(d)} = R_{d+1}^{(d)}$. This struck us as a bit curious – why did the sequence $R_1^{(d)} \subseteq R_2^{(d)} \dots R_d^{(d)} \subseteq R_{d+1}^{(d)} \subseteq \dots$ stabilize at

the d^{th} step? There did not seem to be a natural connection between the number of variables and the number of factors, and this led us to examine his argument more closely. The results of the original paper appear again in the survey paper [2] with some minor typographical errors corrected and in some cases, required conditions were clarified, but we note that there appears to be a minor error in the statement of Lemma 15 of [3] that did not get corrected in the survey paper. Fortunately, this error does not affect the validity of the proof that $R_d^{(d)} = R_{d+1}^{(d)}$. Lemma 15 of [3] states that for each $k = 1, 2, \dots, d-1$, a certain polynomial f_k is congruent modulo $R_1^{(d)}$ to a summation expression. In fact, since it is not known whether or not $R_1^{(d)}$ is an ideal of $k_0\langle X \rangle$, the best that can be said is that f_k is congruent modulo $R_{k-1}^{(d)}$ to the summation expression. Shchigolev's proof that $R_d^{(d)} = R_{d+1}^{(d)}$ only needs that the summation expression that appears in the d^{th} iterate belongs to $R_d^{(d)}$, and since the summation expression is congruent modulo $R_{d-1}^{(d)} \subseteq R_d^{(d)}$ to f_d , and (it is apparent from the definition of f_k) $f_k \in R_k^{(d)}$ for each k , the desired conclusion holds.

In this note, we offer a short proof that over any field k (of any positive characteristic), $R_2^{(d)} = R_3^{(d)}$ for every positive integer d , and that over an infinite field of characteristic $p > 2$, $L_2 = L_3$. Moreover, if the characteristic of k does not divide d , we prove that $R_1^{(d)}$ is an ideal of $k_0\langle X \rangle$ and thus in particular, $R_1^{(d)} = R_2^{(d)}$. Finally, we prove that for every prime $p > 2$, and any field k of characteristic p , $R_1^{(p)} \neq R_2^{(p)}$ and that $L_1 \neq L_2$ (we remark that for an infinite field k , Shchigolev's argument in [3] can be used to imply that $R_1^{(p)} \neq R_2^{(p)}$).

2 $R_2^{(d)} = R_3^{(d)}$

For this section, k is an arbitrary field. The proof of the first result is immediate.

Lemma 2.1. *Let d be a positive integer. Then*

$$S_{d+1}(x) = \sum_{i=1}^{d+1} S_d(x_1, x_2, \dots, \hat{x}_i, \dots, x_{d+1})x_i \quad (1)$$

$$= S_d(x_1, x_2, \dots, x_d)x_{d+1} + \sum_{i=1}^d S_d(x_1, x_2, \dots, x_{d+1}x_i, \dots, x_d) \quad (2)$$

$$= x_{d+1}S_d(x_1, x_2, \dots, x_d) + \sum_{i=1}^d S_d(x_1, x_2, \dots, x_ix_{d+1}, \dots, x_d). \quad (3)$$

Corollary 2.1. *For any $u \in A$, and any positive integer d , $[S_d(x), u] \in R_1^{(d)}$.*

Proof. This follows directly from (2) and (3) of Lemma 2.1. \square

We remark that in [3], Shchigolev proves that if the field is infinite, then for any T -space L , if $v \in L$, then $[v, u] \in L$ for any $u \in A$.

Corollary 2.2. *For any positive integer d , $S_{d+1}(x) \equiv S_d(x)x_{d+1} \equiv x_{d+1}S_d(x) \pmod{R_1^{(d)}}$.*

Proof. This is also immediate from (2) and (3) of Lemma 2.1. \square

Corollary 2.3. *For any positive integer d , $dS_{d+1}(x) \in R_1^{(d)}$.*

Proof. Note that $S_d(x_1, x_2, \dots, x_d) = S_d(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)})$ for any positive integer d and any $\sigma \in \Sigma_d$. By Corollary 2.2, applied $d+1$ times with a different variable pulled out each time, we obtain

$$(d+1)S_{d+1}(x) \equiv \sum_{i=1}^{d+1} S_d(x_1, x_2, \dots, \hat{x}_i, \dots, x_{d+1})x_i \pmod{R_1^{(d)}},$$

and so the result follows from (1) of Lemma 2.1. \square

If the characteristic of k does not divide d , it follows from Corollary 2.3 that $S_{d+1}(x) \in R_1^{(d)}$, and then Corollary 2.2 implies that $R_1^{(d)}$ is an ideal of $k_0\langle X \rangle$. In particular, if the characteristic of k does not divide d , then $R_1^{(d)} = R_2^{(d)}$.

Proposition 2.1. *Let k be a field of characteristic p , and let d be a positive multiple of p . Then $R_2^{(d)} = R_3^{(d)}$.*

Proof. By (1) of Lemma 2.1, $S_d(x)x_{d+1} + \sum_{i=1}^d S_d(x_1, x_2, \dots, \hat{x}_i, \dots, x_{d+1})x_i = S_{d+1}(x)$, and by Corollary 2.2, we have $\sum_{i=1}^d S_d(x_1, x_2, \dots, \hat{x}_i, \dots, x_{d+1})x_i = S_{d+1}(x) - S_d(x)x_{d+1} \in R_1^{(d)}$. Let $u \in R_1^{(d)}$. Then

$$\begin{aligned} uS_d(x_2, \dots, x_{d+1})x_1 + \sum_{i=2}^d uS_d(x_1, \dots, \hat{x}_i, \dots, x_{d+1})x_i \\ = u \sum_{i=1}^d S_d(x_1, x_2, \dots, \hat{x}_i, \dots, x_{d+1})x_i \in R_2^{(d)}. \end{aligned}$$

Since $u \in R_1^{(d)}$, it follows from Corollary 2.1 that for each $i = 2, \dots, d$,

$$uS_d(x_1, \dots, \hat{x}_i, \dots, x_{d+1})x_i \equiv S_d(x_1, \dots, \hat{x}_i, \dots, x_{d+1})x_i u \pmod{R_1^{(d)}},$$

and by two applications of Corollary 2.2, we then obtain

$$\begin{aligned} S_d(x_1, \dots, \hat{x}_i, \dots, x_{d+1})x_i u &\equiv S_{d+1}(x_1, \dots, \hat{x}_i, \dots, x_{d+1}, x_i u) \\ &\equiv S_d(x_2, \dots, \hat{x}_i, \dots, x_{d+1}, x_i u)x_1 \pmod{R_1^{(d)}}. \end{aligned}$$

Thus for any $v \in R_1^{(d)}$, upon replacing x_1 by v we obtain that

$$uS_d(x_2, \dots, x_{d+1})v + \left(\sum_{i=2}^d S_d(x_2, \dots, \hat{x}_i, \dots, x_{d+1}, x_i u) \right) v \in R_2^{(d)},$$

and so $uS_d(x_2, \dots, x_{d+1})v \in R_2^{(d)}$ for all $u, v \in R_1^{(d)}$. It follows that $R_3^{(d)} \subseteq R_2^{(d)}$. \square

3 $L_2 = L_3$

The central idea behind Shchigolev's proof that $L_p = L_{p+1}$ is encapsulated in the following lemma.

Lemma 3.1. *Let k be an infinite field of characteristic $p > 2$, i be any positive integer, and u a multihomogeneous element of L_i . If $uS_p(x)S_p(y) \in L_{i+1}$, where u , $S_p(x)$, and $S_p(y)$ have no generators of $k_0\langle X \rangle$ in common, then $uz_1^p z_2^p \in L_{i+1}$, where $z_1 \neq z_2$ and neither appears in u .*

Proof. For convenience, for any positive integer n , and any subset U of $J_n = \{1, 2, \dots, n\}$, let

$$X_n(U) = \prod_{i=1}^n x_i \Big|_{x_i = \begin{cases} z_1 & i \in U \\ z_2 & i \notin U \end{cases}}.$$

Then $uS_p(x)S_p(y) \in L_{i+1}$ implies that for every j with $1 \leq j \leq p-1$, we have

$$\begin{aligned} u(j!(p-j)!)^2 \sum_{\substack{U \subseteq J_p \\ |U|=j}} X_p(U) \sum_{\substack{U \subseteq J_p \\ |U|=p-j}} X_p(U) \\ = uS_p(\underbrace{z_1, z_1, \dots, z_1}_j, z_2, \dots, z_2) S_p(\underbrace{z_1, z_1, \dots, z_1}_{p-j}, z_2, \dots, z_2) \end{aligned}$$

so $u \sum_{\substack{U \subseteq J_p \\ |U|=j}} X_p(U) \sum_{\substack{U \subseteq J_p \\ |U|=p-j}} X_p(U) \in L_{i+1}$, and thus

$$g = \sum_{j=1}^{p-1} u \sum_{\substack{U \subseteq J_p \\ |U|=j}} X_p(U) \sum_{\substack{U \subseteq J_p \\ |U|=p-j}} X_p(U)$$

is an element of L_{i+1} . On the other hand, since $u \in L_i$, we have $u(z^2)^p \in L_{i+1}$, and so $u(z_1 + z_2)^{2p} \in L_{i+1}$. As k is infinite, h , the sum of all multihomogeneous component of $u(z_1 + z_2)^{2p}$ with degree p for each of z_1 and z_2 , belongs to L_{i+1} . We have

$$h = u \sum_{\substack{U \subseteq J_{2p} \\ |U|=p}} X_{2p}(U) = u(z_1^p z_2^p + z_2^p z_1^p) + g \in L_{i+1},$$

and thus $u(z_1^p z_2^p + z_2^p z_1^p) \in L_{i+1}$. Furthermore, since k is infinite and $z_1^p \in L_1$, we have $[z_1^p, z_2^p] \in L_1$ and thus, since $u \in L_i$, $u(z_1^p z_2^p - z_2^p z_1^p) \in L_{i+1}$. It follows that $2uz_1^p z_2^p \in L_{i+1}$, and since $p > 2$, we obtain $uz_1^p z_2^p \in L_{i+1}$, as required. \square

We need one additional fact.

Lemma 3.2. *Let k be an infinite field, and let L be a T -space of $k_0\langle X \rangle$. For any positive integer d , and any $u \in L$, if $uz^d \in L$, where z is a generator not appearing in u , then $uS_d(x) \in L$.*

Proof. Since k is infinite, we may linearize uz^d with respect to z to obtain that $uS_d(x) \in L$. \square

Lemma 3.2 has the following corollary as an immediate consequence.

Corollary 3.1. *Let k be an infinite field of characteristic p . Then for every positive integer k , $R_k^{(p)} \subseteq L_k$.*

Theorem 3.1. *Let k be an infinite field of characteristic $p > 2$. Then $L_2 = L_3$.*

Proof. By Corollary 3.1, $R_2^{(p)} \subseteq L_2$, and by Proposition 2.1, $R_3^{(p)} = R_2^{(p)}$. Thus $S_p(x)S_p(y)S_p(z) \in R_3 = R_2 \subseteq L_2$. As well, $S_p(x) \in R_1^{(p)} \subseteq L_1$, so by Lemma 3.1, $S_p(x)z_1^p z_2^p \in L_2$. But now, since $S_p(x)z_1^p \in L_2$, we have by Lemma 3.2 that $S_p(x)z_1^p S_p(y) \in L_2$. Since $S_p(x) \in R_1^{(d)} \subseteq L_1 \subseteq L_2$, we have $[S_p(x), z_1^p S_p(y)] \in L_2$ and thus $z_1^p S_p(y)S_p(x) \in L_2$. As $z_1^p \in L_1$, we may apply Lemma 3.1 again to obtain that $z_1^p z_2^p z_3^p \in L_2$. Thus $L_3 \subseteq L_2$. \square

4 A study of R_1^p for prime $p > 2$

In this section, we explore the structure of $R_1^{(p)}$ for an arbitrary prime p and an arbitrary field of characteristic p .

Definition 4.1. *For any positive integer n , and any i with $1 \leq i \leq n+1$, let $M_i^{n+1} = \prod_{j=1}^{n+1} a_j$, where $a_j = x$ if $j = i$, otherwise $a_j = y$.*

The proof of the following result involves an elementary inductive argument based on Pascal's identity, and has been omitted.

Lemma 4.1. *For any integer $n \geq 1$, $[x, \underbrace{y, \dots, y}_n] = \sum_{i=1}^{n+1} (-1)^{i+1} \binom{n}{i-1} M_i^{n+1}$.*

Lemma 4.2. *For any prime p , and any integer i with $0 \leq i \leq p-1$, $\binom{p-1}{i} \equiv (-1)^i \pmod{p}$.*

Proof. The result is immediate for $p = 2$, while for $p > 2$, it follows also from Pascal's identity $\binom{p-1}{i} + \binom{p-1}{i-1} = \binom{p}{i}$, which is zero modulo p if $0 < i < p$. Thus $\binom{p-1}{i} \equiv -\binom{p-1}{i-1}$ for $i = 1, 2, \dots, p-1$. The result follows by induction based on $\binom{p-1}{p-1} = 1$. \square

Corollary 4.1. *Let p be a prime. Then $[x, \underbrace{y, \dots, y}_{p-1}] = \sum_{i=1}^p (-1)^{i+1} \binom{p-1}{i-1} M_i^p$.*

Proof. By Lemma 4.1, we have $[x, \underbrace{y, \dots, y}_{p-1}] = \sum_{i=1}^p (-1)^{i+1} \binom{p-1}{i-1} M_i^p$, and thus by Lemma 4.2, $[x, \underbrace{y, \dots, y}_{p-1}] = \sum_{i=1}^p (-1)^{i+1} (-1)^{i-1} M_i^p$, as required. \square

Proposition 4.1. For any prime p , $S_p(x, \underbrace{y, \dots, y}_{p-1}) = -[x, \underbrace{y, \dots, y}_{p-1}]$.

Proof. $S_p(x, \underbrace{y, \dots, y}_{p-1}) = (p-1)! \sum_{i=1}^p M_i^p$, and so by Wilson's theorem and Corollary 4.1, we obtain the desired result. \square

For any integer $n \geq 2$, let Σ_n^* denote the permutation group on $\{2, 3, \dots, n\}$.

Corollary 4.2. $S_p(x_1, x_2, \dots, x_p) = \sum_{\sigma \in \Sigma_p^*} [x_1, x_{\sigma(2)}, \dots, x_{\sigma(p)}]$ for any prime p .

Proof. Let $u = \sum_{j=2}^p x_j$. Then $S_p(x_1, \underbrace{u, \dots, u}_{p-1}) = -[x_1, \underbrace{u, \dots, u}_{p-1}]$. Upon expansion, we obtain that $(p-1)!S_p(x_1, x_2, \dots, x_p) + \sum_{\sigma \in \Sigma_p^*} [x_1, x_{\sigma(2)}, \dots, x_{\sigma(p)}]$, a sum of monomials each of which depends essentially on x_1, x_2, \dots, x_p , is equal to a sum of monomials each of which is missing at least one of the variables x_2, \dots, x_p . Since the set of all monomials forms a linear basis for $k_0\langle X \rangle$, it follows that $(p-1)!S_p(x_1, x_2, \dots, x_p) + \sum_{\sigma \in \Sigma_p^*} [x_1, x_{\sigma(2)}, \dots, x_{\sigma(p)}] = 0$. Since $(p-1)! = -1$, the result follows. \square

Proposition 4.2. $R_1^{(p)} = \{[x, \underbrace{y, \dots, y}_{p-1}]\}^S$ for any prime p and any field of characteristic p ,

Proof. By Proposition 4.1, $\{[x, \underbrace{y, \dots, y}_{p-1}]\}^S \subseteq R_1^{(p)}$. On the other hand, if we let $u = \sum_{i=2}^p x_i$, then $S_p(x_1, \underbrace{u, \dots, u}_{p-1}) = (p-1)!S_p(x_1, x_2, \dots, x_p) + w$, where w is a sum of monomials each of which is missing at least one variable, and by Proposition 4.1, $S_p(x_1, \underbrace{u, \dots, u}_{p-1}) \in \{[x, \underbrace{y, \dots, y}_{p-1}]\}^S$. Thus $(p-1)!S_p(x_1, x_2, \dots, x_p)$, the part of $S_p(x_1, \underbrace{u, \dots, u}_{p-1})$ which depends essentially on x_1, x_2, \dots, x_p belongs to $\{[x, \underbrace{y, \dots, y}_{p-1}]\}^S$ as well, and so we have $R_1^{(p)} \subseteq \{[x, \underbrace{y, \dots, y}_{p-1}]\}^S$. \square

5 $R_1^{(p)} \neq R_2^{(p)}$ and $L_1 \neq L_2$

Lemma 5.1. For any $x, y \in X$ and monomials u_1, u_2, \dots, u_p in $k_0\langle X \rangle$, the coefficients of $(xy)^p$ and of $(yx)^p$ in $S_p(u_1, \dots, u_p)$ sum to zero.

Proof. We must show that if there exists $\sigma \in \Sigma_p$ such that $\prod_{i=1}^p u_{\sigma(i)} = (xy)^p$ or $(yx)^p$, then the number of such permutations in Σ_p is a multiple of p (actually, this argument does not depend on p being prime). Let $\gamma \in \Sigma_p$ denote the cyclic permutation that sends i to $i+1$ for $1 \leq i \leq p-1$, with p being sent to 1. Then the equivalence relation defined on Σ_p by saying σ is related to τ if and only if there exists an integer t such that $\sigma = \gamma^t \circ \tau$ has as its equivalence classes the right cosets of the cyclic subgroup generated by γ , so each equivalence class has size p . Suppose that $\sigma \in \Sigma_p$ is such that $\prod_{i=1}^p u_{\sigma(i)} = (xy)^p$ or $(yx)^p$. Then $\prod_{i=1}^p u_{\gamma \circ \sigma(i)} = (xy)^p$ or $(yx)^p$ as well (depending on whether $u_{\sigma(p)}$ starts with x or y), and thus for every τ in the equivalence class of σ , we have $\prod_{i=1}^p u_{\tau(i)} = (xy)^p$ or $(yx)^p$. Thus the permutations of u_1, \dots, u_p that produce either $(xy)^p$ or $(yx)^p$ can be partitioned into cells of size p , and so the result follows. \square

Theorem 5.1. *If $p > 2$, then $R_1^p \neq R_2^{(p)}$.*

Proof. Suppose to the contrary that $R_1^p = R_2^{(p)}$. Then in particular,

$$w = S_p(\underbrace{x, x, \dots, x}_{\frac{p+1}{2}}, \underbrace{y, y, \dots, y}_{\frac{p-1}{2}}) S_p(\underbrace{y, y, \dots, y}_{\frac{p+1}{2}}, \underbrace{x, x, \dots, x}_{\frac{p-1}{2}}) \in R_1^{(p)}$$

(note that this is where we use the fact that p is odd). Since $w \in R_1^{(p)}$, w can be written as a linear combination of terms of the form $S_p(u_1, u_2, \dots, u_p)$, $u_1, u_2, \dots, u_p \in k_0\langle X \rangle$. As S_p is multilinear, it follows that w can be written as a linear combination of terms of the form $S_p(u_1, u_2, \dots, u_p)$, where u_1, u_2, \dots, u_p are monomials in $k_0\langle X \rangle$, and thus by Lemma 5.1, the sum of the coefficient of $(xy)^p$ and $(yx)^p$ in w is zero. However, the coefficient of $(xy)^p$ in w is $((\frac{p+1}{2})!(\frac{p-1}{2})!)^2 \not\equiv 0 \pmod{p}$, and the coefficient of $(yx)^p$ in w is 0, which means that the sum of the coefficients of $(xy)^p$ and of $(yx)^p$ in w is not zero, a contradiction. \square

Proposition 5.1. $R_1^{(p)} \neq L_1$.

Proof. Suppose to the contrary that $R_1^{(p)} = L_1$. Then $(xy)^p \in R_1^{(p)}$, which means that $(xy)^p$ can be written as a linear combination of terms of the form (using the fact that S_p is multilinear) $S_p(u_1, u_2, \dots, u_p)$, where u_1, u_2, \dots, u_p are monomials in $k_0\langle X \rangle$. But by Lemma 5.1, in any linear combination of such terms, the sum of the coefficients of $(xy)^p$ and of $(yx)^p$ is zero, while the sum must in fact be 1. Thus we have a contradiction, and the result follows. \square

The following result is well-known.

Lemma 5.2. *For any $u, v \in k_0\langle X \rangle$ and integer i with $1 \leq i \leq p-1$, let*

$$S_p(u, v; i) = S_p(\underbrace{u, u, \dots, u}_i, \underbrace{v, v, \dots, v}_{p-i}).$$

Then $(u + v)^p = u^p + v^p + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} S_p(u, v; i)$.

Theorem 5.2. *For any prime p , $L_1 \neq L_2$.*

Proof. A. V. Grishin proved in [1] that over an infinite field of characteristic 2, the T -space generated by $\{x_1^p, x_1^p x_2^p, \dots\}$ is not finitely based, and thus $L_i \neq L_j$ for any distinct i and j . We now consider the situation when $p > 2$. Suppose to the contrary that $L_1 = L_2$. Since $R_2^{(p)} \subseteq L_2$, we obtain that $R_2^{(p)} \subseteq L_1$ and so in particular, the element w introduced in the proof of Theorem 5.1 belongs to L_1 . But then w is a linear combination of terms of the form u^p , $u \in k_0\langle X \rangle$. By Lemma 5.2, each term of the form u^p , $u \in k_0\langle X \rangle$, can be written as a linear combination of terms of the form v^p or $S_p(u_1, u_2, \dots, u_p)$, where v, u_1, u_2, \dots, u_p are monomials in $k_0\langle X \rangle$. We note that w is multihomogeneous of degree p in each of x and y , and as each expression of the form $S_p(u_1, u_2, \dots, u_p)$ with u_1, u_2, \dots, u_p monomials in $k_0\langle X \rangle$ is multihomogenous, it follows that w can be written as a multihomogeneous linear combination of terms of the form $(xy)^p$, $(yx)^p$, and $S_p(u_1, u_2, \dots, u_p)$, where u_1, u_2, \dots, u_p are monomials in $k_0\langle X \rangle$. Thus we may assume that $w = r(xy)^p + s(yx)^p + z$, where $r, s \in k$ and z is a linear combination of terms of the form $S_p(u_1, u_2, \dots, u_p)$, with u_1, u_2, \dots, u_p monomials in $k_0\langle X \rangle$. Let $\alpha: k_0\langle X \rangle \rightarrow k_1\langle X \rangle$ denote the algebra homomorphism that is determined by sending x to itself, y to 1, and all other elements of X to zero. Since the image of α is contained in the commutative ring $k < \{x\} >$, all elements of $k_0\langle X \rangle$ of the form $S_p(u_1, u_2, \dots, u_p)$ will be mapped to zero. In particular, $\alpha(w) = 0$ and $\alpha(z) = 0$, so we obtain that $(r + s)x^p = 0$ and thus $r + s = 0$. But this, together with Lemma 5.1, means that the sum of the coefficient of $(xy)^p$ and the coefficient of $(yx)^p$ in w is equal to 0. As we have already observed in the proof of Theorem 5.1, this is not the case, and so $L_1 \neq L_2$. \square

References

- [1] A. V. Grishin, *T-spaces with an infinite basis over a field of characteristic 2*, International Conference in Algebra and Analysis Commemorating the Hundredth Anniversary of N. G. Chebotarev, Proceedings, Kazan 5–11, June, 1994, p. 29 (Russian).
- [2] A. V. Grishin and V. V. Shchigolev, *T-spaces and their applications*, Journal of Mathematical Sciences, Vol 134, No. 1, 2006, 1799–1878 (translated from Sovremennaya matematika i-Ee Prilozheniya, Vol. 18, Algèbra, 2004).
- [3] V. V. Shchigolev, *Examples of T-spaces with an infinite basis*, Sbornik Mathematics, Vol 191, No. 3, 2000, 459–476.